Advanced Studies in Contemporary Mathematics
29 (2019), No. 3, pp. 331 – 338
www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2019.29.3.331

A NEW SUMMATION FORMULA FOR ${}_2\psi_2$ BASIC BILATERAL HYPERGEOMETRIC SERIES BY q-EXPONENTIAL OPERATOR TECHNIQUE

S. AHMAD ALI AND ADITYA AGNIHOTRI

ABSTRACT. In the present work by making use of q-exponential operator technique of parameter augmentation, we have obtained a new summation formula for ${}_2\psi_2$ basic bilateral hypergeometric series. We also discuss some interesting applications of our result and deduced some identities of q-gamma, q-beta functions and eta-functions.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 33B15, 33D20, 30D05.

KEYWORDS AND PHRASES. Ramanujan's Sum, q-series, Basic bilateral hypergeometric series, q-gamma function, q-beta function, Dedekind eta-function.

1. INTRODUCTION

In the theory of basic hypergeometric series, one of the most fundamental identity is Ramanujan's ${}_1\psi_1$ sum that can be stated as

$$(1)\qquad {}_1\psi_1(a;b;q,z)=\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n=\frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty},$$

where $|b/a|<|z|<1$. The above sum is regarded as a bilateral extension of well known q-binomial theorem [22]. It was brought to the attention of mathematical community by Hardy who described it as "a remarkable formula with many parameters". The first proof of (1) was given by Hahn [20] and Jackson [25]. Since, then a number of analytical as well as combinatorial proofs have appeared in the literature by Andrews [1,2], Andrews and Askey [3], Askey [6], Ismail [23], Fine [18], Mimachi [27], Venkatachaliengar [31], Corteel and Lovejoy [15], Yee [32], Chan [11] and Chen et. al [14]. The importance of the sum (1) lies in its varied applications and also in the fact that a large number of q-identities of Ramanujan can be derived from it (see [4]).

Recently, Somashekara et.al. [17] have given the following summation formula (2) for basic bilateral hypergeometric series

$$(2)\qquad \sum_{n=-\infty}^{\infty}\frac{(a)_n(bc/azq)_n}{(b)_n(c)_n}z^n =\frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty(c/a)_\infty(bc/azq)_\infty}{(z)_\infty(b)_\infty(c)_\infty(b/az)_\infty(c/az)_\infty(q/a)_\infty}.$$

The proof of (2) is based on the application of an operator $E(b\theta)$ [26, 33] in (1). We noticed that instead of using the operator $E(b\theta)$ in (1) if we use another operator $T(bD_q)$ of Chen and Liu [12, 13], it leads to a new summation identity which is different from (2). In the present communication, we place on record the new summation formula (13) for basic bilateral hypergeometric series ${}_2\psi_2$ and also mention some of its applications.

In order to make the paper readable, we mention some of the notations and definitions [19] that may be required in the next sections. The generalized basic hypergeometric series is defined by

$$ {}_{r+1}\phi_r\left(\begin{array}{cccc} a_1, & a_2 & .. & a_{r+1}; \quad q; z \\ b_1 & b_2 & .. & b_r \end{array}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n..(a_{r+1})_n}{(q)_n(b_1)_n..(b_r)_n} z^n, $$

where $|z| < 1$, $|q| < 1$.

The bilateral basic hypergeometric series is defined by

$$ {}_r\psi_r\left(\begin{array}{cccc} a_1, & a_2 & .. & a_r; \quad q; z \\ b_1 & b_2 & .. & b_r \end{array}\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n(a_2)_n..(a_r)_n}{(q)_n(b_1)_n..(b_r)_n} z^n, $$

where $|\frac{b_1...b_r}{a_1...a_r}| < |z| < 1$, $|q| < 1$ and

$$ (a; q^k)_n = (1-a)(1-aq^k)(1-aq^{2k}).....(1-aq^{k(n-1)}). $$

Whenever k=1, we write

$$ (a; q)_n = (a)_n, $$

$$ (a)_\infty = (a; q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). $$

In this paper, we will use frequently the following:

$$ (a)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \tag{3} $$

$$ (q/a)_n = (-a)^{-n} q^{n(n+1)/2} \frac{(q^{-n}a)_\infty}{(a)_\infty}. \tag{4} $$

The q-analogue of gamma function due to Jackson [24] is

$$ \Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \ 0 < q < 1. \tag{5} $$

and the q-analogue of beta function [5] is defined as

$$ B_q(x, y) = (1-q)\sum_{n=0}^{\infty} q^{nx} \frac{(q^{n+1})_\infty}{(q^{x+y})_\infty}. \tag{6} $$

We also use the relation

$$ B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{7} $$

The q-differential operator or q-derivative is defined as

$$D_q f(a) = \frac{f(a) - f(aq)}{a}. \tag{8}$$

The Leibnitz rule for D_q is the following identity, which is a variation of the q-binomial theorem [29],

$$D_q^n f(a)g(a) = \sum_{k=0}^{n} q^{k(k-n)} \left(\begin{array}{c} n \\ k \end{array} \right) D_q^k f(a) D_q^{n-k} g(q^k a), \tag{9}$$

where D_q^0 is an identity and

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

is the q-binomial coefficient.

Chen and Liu [12, 13] constructed the following q-exponential operator

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}. \tag{10}$$

The following identities hold for this operator

$$T(bD_q) \frac{1}{(at)_\infty} = \frac{1}{(at, bt)_\infty}, \tag{11}$$

$$T(bD_q) \frac{1}{(as, at)_\infty} = \frac{(abst)_\infty}{(as, at, bs, bt)_\infty}. \tag{12}$$

2. Main Results

Theorem 2.1 We have

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n (c)_n}{(b)_n (acz)_n} z^n = \frac{(q)_\infty (b/a)_\infty (az)_\infty (q/az)_\infty (cz)_\infty}{(b)_\infty (q/a)_\infty (z)_\infty (b/az)_\infty (acz)_\infty}. \tag{13}$$

Proof: Ramanujan's ${}_1\psi_1$ summation formula (1) can be written as

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b)_n}{(q/a)_n} (b/az)^n = \frac{(az)_\infty (q)_\infty (q/az)_\infty (b/a)_\infty}{(z)_\infty (b)_\infty (b/az)_\infty (q/a)_\infty}.$$

which on using (3) and (4) becomes

$$\sum_{n=0}^{\infty} \frac{(z)^n}{(b)_n} \{ \frac{1}{(aq^n)_\infty (az)_\infty} \} + \sum_{n=1}^{\infty} \frac{(q/b)_n (b/z)^n}{(-1)^n q^{n(n+1)/2}} \{ \frac{1}{(aq^{-n})_\infty (az)_\infty} \}$$
$$= \frac{(q)_\infty (q/az)_\infty (b/a)_\infty}{(z)_\infty (b)_\infty (b/az)_\infty (q/a)_\infty (a)_\infty}.$$

Applying the operator $T(cD_q)$ on both sides and using the identities (11) and (12), we obtain

$$\sum_{n=0}^{\infty}\frac{(z)^n}{(b)_n}\{\frac{(acq^nz)_\infty}{(aq^n)_\infty(az)_\infty(cq^n)_\infty(cz)_\infty}\}$$
$$+\sum_{n=1}^{\infty}\frac{(q/b)_n(b/z)^n}{(-1)^nq^{n(n+1)/2}}\{\frac{(acq^{-n}z)_\infty}{(aq^{-n})_\infty(az)_\infty(cq^{-n})_\infty(cz)_\infty}\}$$
$$=\frac{(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty(a)_\infty(c)_\infty}.$$

After some simplification, we get

$$\sum_{n=0}^{\infty}\frac{(a)_n(c)_nz^n}{(b)_n(acz)_n}\{\frac{1}{(az)_\infty(cz)_\infty}\}$$
$$+\sum_{n=1}^{\infty}\frac{(a)_{-n}(c)_{-n}}{(b)_{-n}(acz)_{-n}}z^{-n}\{\frac{1}{(az)_\infty(cz)_\infty}\}$$
$$=\frac{(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty(acz)_\infty}.$$

which gives (13).

3. Special Cases

If we take $a=\alpha$, $b=q$, $c=\beta$ and $z=\gamma/\alpha\beta$ in (13), then we obtain

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(q)_n(\gamma)_n}(\gamma/\alpha\beta)^n=\frac{(\gamma/\alpha)_\infty(\gamma/\beta)_\infty}{(\gamma)_\infty(\gamma/\alpha\beta)_\infty}. \tag{14}$$

where $|q|<1$, $|\gamma/\alpha\beta|<1$, which is the q-Gauss summation formula.

Taking $z=q/ac$ in (13), we have

$$\sum_{n=0}^{\infty}\frac{(a)_n(c)_n}{(q)_n(b)_n}(q/ac)^n=\frac{(b/a,q/c,c)_\infty}{(b,q/ac,bc/q)_\infty}.$$

If we let $a\to\infty$

$$\sum_{n=0}^{\infty}\frac{(c)_n(-)^nq^{n(n+1)/2}}{(q)_n(b)_n}(1/c)^n=\frac{(q/c,c)_\infty}{(b,bc/q)_\infty}.$$

On taking $b=q$ and then $c\to\infty$, we obtain

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n^2}=\frac{1}{(q)_\infty}.$$

4. Some q-Gamma and q-Beta Function Identities

1. If $0 < q < 1$, $0 < x, y < 1$ and $0 < x + y < 1$, then

$$(15)\quad B_q(x,y) = \frac{\Gamma_q(1-x+y)\Gamma_q(1+2y)\Gamma_q(x-y)}{\Gamma_q(x)\Gamma_q(1+y)\Gamma_q(2-x+2y)(1-q)^y}\sum_{n=-\infty}^{\infty}\frac{(q^{1-x})_n}{(q^{2-x+2y})_n}q^{yn}.$$

Proof. Putting $a = q^{1-x}$, $z = q^y$ and $b = c = q^{1+y}$ in (13), we have

$$\sum_{n=-\infty}^{\infty}\frac{(q^{1-x})_n}{(q^{2-x+2y})_n}q^{yn} = \frac{(q^{1-x+y})_\infty(q^{x-y})_\infty(q^{1+2y})_\infty(q)_\infty}{(q^{2-x+2y})_\infty(q^x)_\infty(q^{1+y})_\infty}\frac{(q^{x+y})_\infty}{(q^x)_\infty(q^y)_\infty}.$$

Using (5), (6) and (7), we obtain (15).

2. If $0 < q < 1$, $0 < x, y < 1$ and $0 < x + y < 1$, then

$$(16)\quad B_q(x,y) = \frac{\Gamma_q(1-x+y)\Gamma_q(x-y)\Gamma_q(x+y)}{\Gamma_q(x)\Gamma_q(1+y)\Gamma_q(1+y)(1-q)^y}\sum_{n=-\infty}^{\infty}\frac{(q^{1-x})_n(q^x)_n}{(q^{1+y})_n^2}q^{yn}.$$

Proof. Putting $a = q^{1-x}$, $z = q^y$, $b = q^{1+y}$ and $c = q^x$ in (13), we get

$$\sum_{n=-\infty}^{\infty}\frac{(q^{1-x})_n(q^x)_n}{(q^{1+y})_n^2}q^{yn} = \frac{(q^{1-x+y})_\infty(q^{x-y})_\infty(q^{x+y})_\infty(q)_\infty}{(q^{1+y})_\infty(q^x)_\infty(q^{1+y})_\infty}\frac{(q^{x+y})_\infty}{(q^x)_\infty(q^y)_\infty}.$$

Using (5), (6) and (7), we obtain (16).

3. If $0 < q < 1$, $0 < x, y < 1$ and $0 < x + y < 1$, then

$$(17)\quad B_q(x,y) = \frac{\Gamma_q(1-x-y)\Gamma_q(y)}{(1-q)^x}\left\{\frac{\Gamma_q(x-y+1)}{\Gamma_q(1-y)\Gamma_q(1+x)}\right\}^2\sum_{n=-\infty}^{\infty}\frac{(q^y)_n(q^{1-y})_n}{(q^{1+x})_n^2}q^{xn}.$$

Proof. Putting $a = q^y$, $z = q^x$, $b = q^{1+x}$ and $c = q^{-y+1}$ in (13), we get

$$\sum_{n=-\infty}^{\infty}\frac{(q^y)_n(q^{1-y})_n}{(q^{1+x})_n^2}q^{xn} = \frac{(q^{x+y})_\infty(q^{x-y+1})_\infty(q^{1-x-y})_\infty(q)_\infty(q^{1+x-y})_\infty}{(q^x)_\infty(q^{1-y})_\infty(q^{1-y})_\infty(q^{1+x})_\infty(q^{1+x})_\infty}.$$

Using (5), (6) and (7), we obtain (17).

4. If $0 < q < 1$, $0 < x, y < 1$, then

$$(18)\quad B_q^2(x,y) = \frac{\Gamma_q(x)\Gamma_q(y-x)\Gamma_q(1+x-y)}{\Gamma_q(y)\Gamma_q(1+x)(1-q)^{y-1}}\sum_{n=-\infty}^{\infty}\frac{(q^{-x})_n(q^x)_n}{(q^y)_n^2}q^{yn}.$$

Proof. Putting $a = q^y$, $z = q^x$, $b = q^{1+x}$ and $c = q^{-y+1}$ in (13), we get

$$\sum_{n=-\infty}^{\infty}\frac{(q^{-x})_n(q^x)_n}{(q^y)_n^2}q^{yn} = \frac{(q^{y-x})_\infty(q^{x+y})_\infty(q^{1+x-y})_\infty(q)_\infty(q^{y+x})_\infty}{(q^y)_\infty(q^{1+x})_\infty(q^x)_\infty(q^y)_\infty(q^y)_\infty}.$$

Using (5), (6) and (7), we obtain (18).

5. If $0 < q < 1$, $0 < x, y < 1$ and $0 < x + y < 1$, then

$$\sum_{n=0}^{\infty} \frac{(q^{1-x-y})_n (q^x)_n}{(q)_n^2} q^{yn} = \frac{\Gamma_q(y)}{\Gamma_q(x+y)\Gamma_q(1-x)}. \tag{19}$$

Proof. Putting $a = q^{1-x-y}$, $b = q$, $c = q^x$ and $z = q^y$ in (13), we get

$$\sum_{n=0}^{\infty} \frac{(q^{1-x-y})_n (q^x)_n}{(q)_n^2} q^{yn} = \frac{(q^{1-x})_\infty (q^{x+y})_\infty}{(q^y)_\infty (q)_\infty}.$$

Using (5), (6) and (7), we obtain (19).
If we take $q \to 1$ in (19), then we obtain

$$\frac{\Gamma(y)}{\Gamma(1-x)\Gamma(x+y)} = \sum_{n=0}^{\infty} \frac{(1-x-y)_n (x)_n}{(n!)^2}.$$

6. If $0 < q < 1$, $0 < x, y < 1$, then

$$B_q(x, y) = \frac{\Gamma_q(x)}{\Gamma_q(x+1)} \sum_{n=0}^{\infty} \frac{(q^x)_n (q^{1-y})_n}{(q)_n (q^{x+1})_n} q^{yn}. \tag{20}$$

Proof. Putting $a = q^x$, $b = q$, $c = q^{1-y}$ and $z = q^y$ in (13), we get

$$\sum_{n=0}^{\infty} \frac{(q^x)_n (q^{1-y})_n}{(q)_n (q^{x+1})_n} q^{yn} = \frac{(q^{x+y})_\infty (q)_\infty}{(q^{x+1})_\infty (q^y)_\infty}.$$

Using (5), (6) and (7), we obtain (20).
If we take $q \to 1$ in (20), then we obtain

$$B(x, y) = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(x)_n (1-y)_n}{n!(x+1)_n}.$$

7. If $0 < q < 1$, $0 < x, y < 1$, then

$$B_q^2(x, y) = \frac{\Gamma_q(x)\Gamma_q(y-x)\Gamma_q(1+x-y)}{\Gamma_q^2(1+x)(1-q)^x} \sum_{n=-\infty}^{\infty} \frac{(q^{1-y})_n (q^y)_n}{(q^{x+1})_n^2} q^{xn}. \tag{21}$$

Proof. Putting $a = q^{1-y}$, $z = q^x$, $b = q^{1+x}$ and $c = q^y$ in (13), we get

$$\sum_{n=-\infty}^{\infty} \frac{(q^{1-y})_n (q^y)_n}{(q^{1+x})_n^2} q^{xn} = \frac{(q^{1-y+x})_\infty (q^{x+y})_\infty (q^{y-x})_\infty (q)_\infty (q^{x+y})_\infty}{(q^x)_\infty (q^y)_\infty (q^y)_\infty (q^{1+x})_\infty (q^{1+x})_\infty}.$$

Using (5), (6) and (7), we obtain (21).

5. Some Eta-function Identities

The Dedekind eta-function is given by

$$\eta(\tau) = e^{\pi\iota\tau/12} \prod_{n=1}^{\infty} (1 - e^{2n\pi\iota\tau}) = q^{1/24}(q; q)_\infty, \tag{22}$$

where $q = e^{2\pi\iota\tau}$ and $Im\tau > 0$.

The identity (13) can be used to established a number of eta-function identities. We mention here only two of such identities

$$\frac{\eta(2\tau)}{\eta(\tau)} = \frac{q^{1/24}}{(1-q^2)(-q^2;q^2)_\infty(q^5;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(-q^6;q^2)_n}q^n. \tag{23}$$

$$\frac{\eta^2(\tau)}{\eta^4(2\tau)} = \frac{(1-q)}{q^{1/4}(1+q)}\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2}{(q^2;q^2)_n(q^4;q^2)_n}q^{2n}. \tag{24}$$

To prove (23), choosing $a=-q^{1/2}$, $z=q^{1/2}$, $b=q$ and $c=q^2$ in (13), we obtain

$$\sum_{n=0}^{\infty}\frac{(-q^{1/2})_n(q^2)_n}{(q)_n(-q^3)_n}q^{n/2} = \frac{(-q)_\infty(q^{5/2})_\infty}{(q^{1/2})_\infty(-q^3)_\infty}.$$

A change of base q to q^2 gives

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(q^4;q^2)_n}{(q^2;q^2)_n(-q^6;q^2)_n}q^n = \frac{(-q^2;q^2)_\infty(q^5;q^2)_\infty}{(q;q^2)_\infty(-q^6;q^2)_\infty}.$$

After some simplification and using (22), we get (23).

Similarly, putting $a=q^{1/2}$, $z=q$, $b=q$ and $c=q^{1/2}$ in (13) and then changing q to q^2, we obtain

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(q;q^2)_n}{(q^2;q^2)_n(q^4;q^2)_n}q^{2n} = \frac{(q^3;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty(q^4;q^2)_\infty}.$$

which after simplification with use of (22) gives (24).

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S. Ahmad Ali, Department of Mathematics and Computer Science, School of Basic Sciences, Babu Banarasi Das University, Lucknow 226028 India.
E-mail address: ali.sahmad@yahoo.com

Aditya Agnihotri, Department of Mathematics and Computer Science, School of Basic Sciences, Babu Banarasi Das University, Lucknow 226028 India.
E-mail address: aditya_agnihotri20@rediffmail.com